Exam Lie Groups in Physics

Date	February 7, 2017
Room	5419.0112
Time	18:30 - 21:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	6	2a)	10	3a)	10	4a)	6
1b)	10	2b)	9	3b)	6	4b)	9
1c)	6	2c)	6			4c)	6
						4d)	6

$$\text{Result } = \frac{\sum \text{points}}{10} + 1$$

Problem 1

(a) Consider the sets of complex numbers C and nonzero complex numbers $\mathsf{C}\backslash\{0\}$. Indicate the composition laws under which these sets form Lie groups. Explain your answers and explain why division cannot be a composition law.

(b) Consider the group $SU(1,1)$ formed by the complex 2×2 matrices U with determinant equal to 1 that satisfy

$$U^{\dagger} = gU^{-1}g^{-1} \quad \text{with} \quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Give a parametrization of all the elements of $SU(1,1)$ and show that the set of those matrices indeed form a group under matrix multiplication.

(c) Show that $SU(1,1)/SO(1,1)$ does not form a group, where $SO(1,1)$ is the group formed by the subset of real matrices in $SU(1,1)$.

Problem 2

Consider the lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n\times n$ matrices with determinant equal to 1.

(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$

$$\yng(2,2,1) \otimes \yng(2,1)$$

into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(2)$ and $su(3)$. Indicate complex conjugate irreps whenever appropriate.

(c) Use the decomposition (Clebsch-Gordan series) of the direct product between the defining representation $\boldsymbol{n}$ of $su(n)$ and its complex conjugate $\boldsymbol{n}^*$ to show that the adjoint representation is real.

Problem 3

Consider the Lie group $SU(2)$ of unitary 2×2 matrices with determinant equal to 1 and the Lie group $SO(3)$ of orthogonal 3×3 matrices with determinant equal to 1. Consider the mapping ϕ from $SU(2)$ onto $SO(3)$ given by:

$$\phi(U)_{ij} = \frac{1}{2}\mathrm{Tr}\left(\sigma_i U \sigma_j U^{-1}\right),$$

where the Pauli matrices σ_i $(i = 1, 2, 3)$ are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Use the completeness relation $\vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}$ to show that the mapping ϕ is a homomorphism.

(b) Identify the kernel of the mapping ϕ and explain its relevance to physics.

Problem 4

Consider the Lie group $O(4)$ of orthogonal transformations in 4-dimensional Euclidean space, which is for instance the symmetry group of the Higgs field potential in the Standard Model of elementary particles and of the Hamiltonian of the nonrelativistic hydrogen atom, and its subgroup $SO(4)$ of elements with determinant equal to 1.

(a) Determine the dimension of $O(4)$, by considering the constraints that need to be satisfied by its generators M_{ab} in the defining representation $(a, b = 1, 2, 3, 4)$.

The generators span the Lie algebra $o(4)$ of $O(4)$. The generators defined as $L_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and $K_i = M_{i4}$ $(i, j, k = 1, 2, 3)$, satisfy the following algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = i\epsilon_{ijk}L_k.$$

(b) Show explicitly that $o(4) \cong so(4)$ and $so(4) \cong so(3) \oplus so(3)$.

(c) Write down the expressions for the two Casimir operators of $o(4)$ in terms of the generators.

(d) Explain that $SO(4)/SO(3)$ is a group and is isomorphic to $SO(3)$ (Hint: $SO(4)$ is compact and connected, hence all elements can be written as a single exponential of Lie algebra elements).